Ex.10　解: 二次型$f(x) = 2x_1x_2 + 2x_1x_3 - 2x_2x_3$ 的矩阵

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}.$$

(1) 求二次型f 的标准形.

计算矩阵A 的特征值. 由于

$$|\lambda E - A| = \begin{vmatrix} \lambda & -1 & -1 \\ -1 & \lambda & 1 \\ -1 & 1 & \lambda \end{vmatrix} = (\lambda - 1)^2(\lambda + 2),$$

所以, 矩阵A 的特征值为$\lambda_1 = \lambda_2 = 1$, $\lambda_3 = -2$.

当$\lambda = 1$ 时, 解齐次线性方程组$(E - A)x = 0$. 由

$$E - A = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

得基础解系

$$\xi_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \ \xi_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

正交化. 取$\eta_1 = \xi_1$,

$$\eta_2 = \xi_2 - \frac{\eta_1^T \xi_2}{\eta_1^T \eta_1} \eta_1 = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix}.$$

当$\lambda = -2$ 时, 解齐次线性方程组$(-2E - A)x = 0$. 由

$$\begin{aligned} -2E - A &= \begin{pmatrix} -2 & -1 & -1 \\ -1 & -2 & 1 \\ -1 & 1 & -2 \end{pmatrix} \to \begin{pmatrix} 0 & 3 & -3 \\ -1 & -2 & 1 \\ 0 & 3 & -3 \end{pmatrix} \\ &\to \begin{pmatrix} -1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

得基础解系

$$\xi_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

特征向量η_1, η_2, ξ_3 是两两正交的, 还需要单位化.

$$\tilde{\eta}_1 = \frac{\eta_1}{||\eta_1||} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \ \tilde{\eta}_2 = \frac{\eta_2}{||\eta_2||} = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{pmatrix}, \ \tilde{\eta}_3 = \frac{\xi_3}{||\xi_3||} = \begin{pmatrix} -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}.$$

于是, 可求出正交矩阵

$$P = (\tilde{\eta}_1, \tilde{\eta}_2, \tilde{\eta}_3) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}.$$

使得$P^TAP = \Lambda$, 其中,

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

令正交变换$x = Py$, 那么, 二次型f 的标准形为

$$f(x) = x^TAx = (Py)^TA(Py) = y^T(P^TAP)y = y^T\Lambda y = y_1^2 + y_2^2 - 2y_3^2.$$

(2) 求二次型f 在$||x|| = 1$ 下的最小值问题可表示为

$$\begin{aligned} \min \quad & f(x), \\ s.t. \quad & ||x|| = 1. \end{aligned}$$

因为正交变换不改变向量长度, 因此上述问题等价于

$$\begin{aligned} \min \quad & y_1^2 + y_2^2 - 2y_3^2, \\ s.t. \quad & y_1^2 + y_2^2 + y_3^2 = 1. \end{aligned}$$

进一步, 该问题等价于

$$\min 1 - 3y_3^2, \ s.t. \ y_3^2 = 1 - y_1^2 - y_2^2 \leq 1.$$

即

$$\min 1-3y_3^2,\ s.t.\ y_3\in[-1,1].$$

显然, 当$y_3=\pm1$ 时二次型f 达到最小值-2.

类似地, 当$y_3=0$ 时二次型f 达到最大值1.